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Strong Unicity of Order 2 in $C(T)$

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1. INTRODUCTION AND PRELIMINARIES

Let M be a nonempty closed subset of a Banach space X. Then an element m in M is called a *strongly unique best approximation* of order $q \geqslant 1$ to an element x in X, if there exists a constant $c = c_M(x) > 0$ such that

$$\|x-m\|^q \leqslant \|x-y\|^q - c\,\|m-y\|^q \tag{1.1}$$

for all y in M. Clearly, a strongly unique best approximation m of order q is the unique best approximation in M to the element x. In recent papers [7, 11–14] we have shown that the converse statement is also true for any sun M (in particular, for any convex subset M) of Lebesgue spaces L_p, Sobolev spaces $W^{k,p}$, Hardy spaces H^p, $L_q(L_p)$-spaces, and some other spaces, where $1 < p < \infty$, $k \geqslant 0$, and $q = \max(2, p)$. Moreover, for all these spaces there exists a constant $c_p > 0$ such that $c_M(x) \geqslant c_p$ for all elements x and suns M. The same result is also true [13] when X is a super-reflexive space with a properly chosen norm equivalent to the original norm in X.

In this paper we shall study the existence of strongly unique best approximations of order 2 in the Banach space $X = C(T)$ of all real-valued, or complex-valued, continuous functions defined on a compact Hausdorff space T endowed with the uniform norm. Note that if an element m is a strongly unique best approximation (i.e., a strongly unique best approximation of order 1) in M to an element $x \in X$, then by (1.1) and the triangle inequality for the norm we have

$$\begin{aligned}\|x-y\|^2 - \|x-m\|^2 &= (\|x-y\| - \|x-m\|)(\|x-y\| + \|m-x\|)\\ &\geqslant c\,\|m-y\|^2.\end{aligned}$$

This means that the element m is also a strongly unique best approximation of order 2 to the element x with the same positive constant $c = c_M(x)$. Therefore, we shall restrict our investigations of strong unicity of

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order 2 to subsets M of $C(T)$ such that strong unicity fails for some elements x in $C(T)$.

It should be remarked that strong unicity of order 2 is very useful to prove a Hölder continuity of metric projections and to establish a rate of convergence of numerical algorithms for computing best approximations. Indeed, let E_M be the set of all elements $x \in X$ having a best approximation m in M, i.e., such that

$$\|x-m\| = \operatorname{dist}(x, M) := \inf_{y \in M} \|x-y\|. \tag{1.2}$$

Denote by P_M the metric projection of E_M into $(\mathfrak{M}, \rho)$ defined by

$$P_M x = \{\text{the set of all best approximations in } M \text{ to } x\},$$

where $(\mathfrak{M}, \rho)$ is the metric space of all nonempty closed bounded subsets of M with the Hausdorff metric

$$\rho(U, V) = \max\{\sup_{u \in U} \operatorname{dist}(u, V), \sup_{v \in V} \operatorname{dist}(v, U)\};\ U, V \in \mathfrak{M}.$$

Moreover, let SU_M be the set of all elements $x \in X$ having a strongly unique best approximation m of order 2 in M, i.e., such that

$$\|x-m\|^2 \leqslant \|x-y\|^2 - c\,\|m-y\|^2 \tag{1.3}$$

for all $y \in M$, where $c = c_M(x)$ is a positive constant independent of y. Clearly, we have $E_M \supset SU_M \supset M$.

THEOREM 1.1. *If $x \in SU_M$ and $0 \in M$, then the metric projection P_M satisfies the local Hölder condition*

$$\rho(P_M x, P_M z) \leqslant d\,\|x-z\|^{1/2}$$

for all $z \in E_M$ such that $\|z\| \leqslant K$, where K is an arbitrary positive constant and the constant d is equal to

$$d = 2[(K + \|x\|)/c_M(x)]^{1/2}.$$

Proof. Let $m = P_M x$ and $u \in P_M z$. Then using (1.2), (1.3), the triangle inequality for the norm, and the fact that $0 \in M$ we obtain

$$\begin{aligned} c_M(x)\,\|m-u\|^2 &\leqslant \|x-u\|^2 - \|x-m\|^2 \leqslant 2(\|x-u\| + \|x-m\|)\,\|x-z\| \\ &\leqslant 4(K + \|x\|)\,\|x-z\|. \end{aligned}$$

Taking the supremum over u of the left-hand side we finish the proof. ∎

Now suppose that a numerical algorithm produces a sequence $\{m_k\}$ in M such that

$$e_k := \|x - m_k\|^2 \to e := [\text{dist}(x, M)]^2 \qquad \text{as } k \to \infty.$$

Then the additional assumption that $x \in SU_M$ enables us to insert $y = m_k$ into inequality (1.3) and get

THEOREM 1.2. *The sequence* $\{m_k\}$ *converges to* $m = P_M x$ *and the estimate*

$$\|m - m_k\|^2 \leqslant (e_k - e)/c_M(x)$$

holds for all k.

2. LINEAR COMPLEX APPROXIMATION

Throughout this section we assume that M is an n-dimensional subspace of the complex Banach space $C(T)$, where T is a compact Hausdorff space which consists at least $n+1$ distinct points. By local compactness of M we have $E_M = C(T)$. It is well known [9, Theorem 6.2] that an element m is a best approximation in M to $x \in C(T)$ if and only if there exist points $(t_j)_1^k$ $(1 \leqslant k \leqslant 2n+1)$ in the set

$$\text{ext}(x - m) = \{t \in T\colon |x(t) - m(t)| = \|x - m\|\}$$

and real positive numbers $(\alpha_j)_1^k$ such that $\sum_{j=1}^k \alpha_j = 1$ and

$$\sum_{j=1}^{k} \alpha_j \overline{(x(t_j) - m(t_j))}\, y(t_j) = 0 \qquad \text{for all } y \in M. \tag{2.1}$$

Additionally, if M is a Haar subspace (i.e., if an element $y \in M \backslash \{0\}$ has at most $n-1$ zeroes in T) then we have $k \geqslant n+1$ [9, Theorem 6.3]. In this case we immediately conclude that the function

$$|y|_m := \left(\sum_{j=1}^{k} \alpha_j\, |y(t_j)|^2\right)^{1/2}; \qquad y \in M, \tag{2.2}$$

is a norm on M. Since all norms on a finite dimensional space are equivalent [2, Corollary 3, p. 245], it follows that there exists a constant $c = c_M(x) > 0$ such that

$$|y|_m^2 \geqslant c\, \|y\|^2 \tag{2.3}$$

for all $y \in M$, where M is a Haar subspace of $C(T)$. Now we establish an interesting theorem which is given in [10, Theorem 2.4.5]. A proof of this theorem is presented, since it is much simpler than the original proof. Moreover, the strong unicity constant $c = c_M(x)$ given below is better than the constant obtained in [10].

THEOREM 2.1. *Let m be a best approximation in a Haar subspace M of $C(T)$ to an element $x \in C(T)$. Then the inequality*

$$\|x - m\|^2 \leqslant \|x - y\|^2 - c\,\|m - y\|^2$$

holds for all $y \in M$, where the positive constant $c = c_M(x)$ is defined as in (2.3).

Proof. Let α_j and t_j be as in (2.1). Since

$$\|x - y\|^2 \geqslant |(x - m)(t_j) + (m - y)(t_j)|^2 = \|x - m\|^2 + |(m - y)(t_j)|^2 + 2\,\mathrm{Re}[\overline{(x - m)(t_j)}(m - y)(t_j)]$$

for all j and $y \in M$, we can multiply the obtained inequalities by α_j, sum up them over j, and use (2.1)–(2.3) and the fact that $\sum \alpha_j = 1$ to complete the proof. ∎

By this theorem and Theorem 1.1 we immediately get

COROLLARY 2.1. *Let x be a function in $C(T)$, and let the positive constant $c = c_M(x)$ be as in (2.3). Then the metric projection P_M of $C(T)$ onto a Haar subspace M of $C(T)$ satisfies the local Hölder condition*

$$\|P_M x - P_M z\| \leqslant d\,\|x - z\|^{1/2}$$

for all $z \in C(T)$ such that $\|z\| \leqslant K$, where K is an arbitrary positive constant and the constant d is equal to

$$d = 2[(K + \|x\|)/c]^{1/2}.$$

Further, Theorems 2.1 and 1.2 point out that any numerical algorithm for computing best approximations in a Haar subspace M of the complex Banach space $C(T)$ should minimize on M the convex functional

$$f(y) := \|x - y\|^2 = \max_{t \in T} |x(t) - y(t)|^2$$

instead of the functional $g(y) = \|x - y\|$, $y \in M$. This is very favourable, since $f(y)$ can be computed with smaller computational effort than $g(y)$.

3. Monotone Approximation

For given integers $1 \leqslant k_1 < k_2 < \cdots < k_p \leqslant n$ $(p \geqslant 1)$ and signs $\varepsilon_j = \pm 1$ $(j = 1, ..., p)$, we denote by M_p the positive convex cone in the real Banach space $C[a, b]$, with the uniform norm $\|\cdot\|$, defined by

$$M_p = \{y \in \pi_n : \varepsilon_j y^{(k_j)}(t) \geqslant 0 \text{ for } a \leqslant t \leqslant b \text{ and } j = 1, ..., p\}, \tag{3.1}$$

where π_n is the subspace of all real algebraic polynomials in $C[a, b]$ of degree n or less. Now, let x be a function in $C[a, b]$. Denote by m a best approximation in M_p to x. Such an approximation exists and is unique (see [5, 6]). By a theorem of G. G. Lorentz and K. L. Zeller [5, Theorem 1] the polynomial $m \in M_p$ is a best approximation in M_p to the function $x \in C[a, b]$ if and only if there exist at most $n+2$ points t_i $(i = 1, ..., l)$ and $s_{j\mu}$ $(j = 1, ..., p;\ \mu = 1, ..., r_j)$ in $[a, b]$ and numbers $\alpha_i > 0$ and $\beta_{j\mu} > 0$ which satisfy

$$|(x-m)(t_i)| = \|x - m\| \qquad \text{and} \qquad m^{(k_j)}(s_{j\mu}) = 0 \qquad \text{for all } i, j, \mu, \tag{3.2}$$

and

$$\sum_{i=1}^{l} \alpha_i \sigma(t_i)\, y(t_i) + \sum_{j=1}^{p} \varepsilon_j \sum_{\mu=1}^{r_j} \beta_{j\mu} y^{(k_j)}(s_{j\mu}) = 0 \tag{3.3}$$

for all $y \in \pi_n$, where $\sigma(t_i) = \operatorname{sgn}(x-m)(t_i)$. In particular, by (3.3) we have

$$\sum_{i=1}^{l} \alpha_i \sigma(t_i)\, y(t_i) = 0$$

for every $y \in \pi_{k_1 - 1}$. This in conjunction with the fact that π_{k_1-1} is a Haar subspace of dimension k_1 implies that $l \geqslant k_1 + 1$ (see [9, Theorem 6.3]). Therefore, one can assume that

$$\sum_{i=1}^{l} \alpha_i = 1. \tag{3.4}$$

Define the seminorm

$$|z|_m = \left\{ \sum_{i=1}^{l} \alpha_i\, |z(t_i)|^2 + \frac{1}{2\lambda} \sum_{j=1}^{p} \sum_{\mu=1}^{r_j} \beta_{j\mu}\, |z^{(k_j)}(s_{j\mu})|^2 \right\}^{1/2} \tag{3.5}$$

on π_n, where

$$\lambda = \max_{1 \leqslant j \leqslant p} [n \cdots (n - k_j + 1)]^2\, [2/(b-a)]^{k_j}. \tag{3.6}$$

LEMMA 3.1. *There exists a constant* $c_1 > 0$ *such that*

$$|z|_m^2 \geqslant c_1 \|z\|^2$$

for all z in the set $Z := \{\alpha(m-y) \colon \alpha \geqslant 0,\ y \in M_p\}$.

Proof. If there exists a polynomial $z = \alpha m - \alpha \hat{y} \in Z \backslash \{0\}$ such that $|z|_m = 0$, then it follows from (3.2), (3.5), and Formula (1.7) in G. G. Lorentz and K. L. Zeller [5] that the Birkhoff interpolation problem

$$\begin{aligned} y(t_i) &= \alpha m(t_i) && (i = 1, ..., l), \\ y^{(k_j)}(s_{j\mu}) &= 0 && (j = 1, ..., p;\ \mu = 1, ..., r_j), \\ y^{(k_j+1)}(s_{j\mu}) &= 0 && (a < s_{j\mu} < b;\ j = 1, ..., p;\ \mu = 1, ..., r_j) \end{aligned}$$

has two different solutions $y = \alpha m$ and $y = \alpha \hat{y}$, which is impossible by Lemma 2.2 of R. A. Lorentz [6] or D. Schmidt [8]. Therefore, we have $|z|_m \neq 0$ for all $z \neq 0$ in Z. Since the nonempty set

$$S(Z) = \{z \in Z \colon \|z\| = 1\}$$

is compact, it follows that

$$c_1 := \inf_{z \in S(Z)} |z|_m^2 > 0. \tag{3.7}$$

Hence we get

$$|z|_m^2 = \|z\|^2\, |z/\|z\|\,|_m^2 \geqslant c_1 \|z\|^2$$

for all $z \neq 0$ in Z. ∎

Now we show that strong unicity of order 2 of best approximations in M_p follows easily from the theorem of G. G. Lorentz and K. L. Zeller and Lemma 3.1.

THEOREM 3.1. *Let m denote a best approximation in* M_p *to a function* $x \in C[a, b]$. *Then there exists a constant* $c > 0$ *such that*

$$\|x-m\|^2 \leqslant \|x-y\|^2 - c\,\|m-y\|^2 \tag{3.8}$$

for all $y \in M_p$.

Proof. Let y be a polynomial in M_p. If $\|m-y\| \geqslant 4\,\|x-m\|$, then by the triangle inequality for the norm we obtain

$$\begin{aligned} \|x-y\|^2 - \|x-m\|^2 &\geqslant (\|x-m\| - \|m-y\|)^2 - \|x-m\|^2 \\ &= \|m-y\|\,(\|m-y\| - 2\,\|x-m\|) \geqslant \tfrac{1}{2}\,\|m-y\|^2. \end{aligned}$$

Otherwise, in view of Markoff's inequality [1, pp. 91, 94], we have

$$\max_{j,\mu} |(m-y)^{(k_j)}(s_{j\mu})| \leqslant \lambda \|m-y\| < 4\lambda \|x-m\|, \tag{3.9}$$

where λ is as in (3.6). On the other hand, if we multiply the inequalities

$$\|x-y\|^2 \geqslant \|x-m\|^2 + |(m-y)(t_j)|^2 + 2(x-m)(t_j)(m-y)(t_j), \qquad j=1, ..., l,$$

by α_j, sum up them over j, and use (3.1)–(3.4) then we get

$$\|x-y\|^2 \geqslant \|x-m\|^2 + \sum_{i=1}^{l} \alpha_i |(m-y)(t_i)|^2$$

$$+2 \sum_{j=1}^{p} \sum_{\mu=1}^{r_j} \beta_{j\mu} \|x-m\| \, |(m-y)^{(k_j)}(s_{j\mu})|.$$

This in conjunction with (3.5) and (3.9) implies that

$$\|x-y\|^2 \geqslant \|x-m\|^2 + |m-y|_m^2.$$

Hence one can apply Lemma 3.1 to derive inequality (3.8) with the positive constant $c=c_1$ defined as in (3.7). Consequently, the constant $c=\min\{\frac{1}{2}, c_1\}$ independent of the y's is admissible in (3.8). ▮

Theorem 3.1 is essentially due to D. Schmidt [8] and B. L. Chalmers and G. D. Taylor [3], who proved that, for every $\varepsilon>0$, there exists a constant $c>0$ such that the inequality

$$\|x-m\| \leqslant \|x-y\| - c\,\|m-y\|^2$$

holds for all $y \in M_p$ with $\|m-y\| \leqslant \varepsilon$. Indeed, one can show that this inequality is equivalent to inequality (3.8). The following corollary follows directly from Theorems 1.1 and 3.1.

Corollary 3.1. *Let x be a function in $C[a, b]$, and let the positive constant c be defined as in Theorem 3.1. Then the metric projection $P\colon C[a, b] \to M_p$ satisfies the local Hölder condition*

$$\|Px-Pz\| \leqslant d\,\|x-z\|^{1/2}$$

for all $z \in C[a, b]$ such that $\|z\| \leqslant K$, where K is an arbitrary positive constant and the constant d is equal to

$$d = 2[(K + \|x\|)/c]^{1/2}.$$

It should be noticed that this corollary was proved by D. Schmidt

[8, Theorem 4.2] under the additional assumption deg $Px \geqslant k_p$. Moreover, an immediate consequence of Corollary 3.1 is the continuity of the metric projection P at each point $x \in C[a, b]$ with respect to the uniform topology in $C[a, b]$, which was proved in [8, Theorem 4.1].

4. Complex Approximation with Hermite Interpolatory Constraints

A detailed study of real approximation with Hermite interpolatory constraints was done by H. L. Loeb *et al.* [4]. In this section we consider approximation of this kind in the Banach space $C(T)$ of all complex-valued continuous functions defined on a compact subset T of the complex plane. We shall assume below that T consists of at least $n+1$ distinct points. Let $\pi_n = \pi_n(T)$ denote the $(n+1)$-dimensional subspace of all complex-valued algebraic polynomials on T of degree n or less. Moreover, let $S = \{s_1, ..., s_p\}$ be a nonempty subset of fixed points in T, let $(n_j)_1^p$ be a given sequence of positive integers with

$$r := n_1 + n_2 + \cdots + n_p \leqslant n,$$

and let (a_{vj}), $1 \leqslant v \leqslant p$ and $1 \leqslant j \leqslant n_v - 1$, be a given array of complex numbers. If x is a function in $C(T)$, then we define the convex subset $M[x]$ of $C(T)$ by

$$M[x] = \{ y \in \pi_n : y^{(j)}(s_v) = a_{vj}; \qquad 1 \leqslant v \leqslant p, 0 \leqslant j \leqslant n_v - 1 \},$$

where $a_{v0} = x(s_v)$ for all v. Clearly, a best approximation m in $M[x]$ to x exists.

THEOREM 4.1. *Let m be a best approximation in $M[x]$ to $x \in C(T)$. Then there exists a constant $c > 0$ such that*

$$\|x - m\|^2 \leqslant \|x - y\|^2 - c\,\|m - y\|^2$$

for all $y \in M[x]$.

Proof. It is clear that 0 is a best approximation in the $(n+1-r)$-dimensional subspace $M := M[x] - m$ of π_n to the function $x - m$. Therefore, by (2.1) there exist positive numbers $\alpha_1, ..., \alpha_k$ ($\sum \alpha_v = 1$, $1 \leqslant k \leqslant 2(n+1-r)+1$) and points $t_1, ..., t_k$ in $\operatorname{ext}(x-m) \subset T \setminus S$ such that

$$\sum_{j=1}^{k} \alpha_j \overline{(x-m)(t_j)}\,(y-m)(t_j) = 0$$

for all $y \in M[x]$. Hence, as in the proof of Theorem 2.1, we get

$$\|x-y\|^2 \geqslant \|x-m\|^2 + |y-m|_m^2$$

for all $y \in M[x]$, where

$$|z|_m = \left(\sum_{j=1}^{k} \alpha_j \, |z(t_j)|^2 \right)^{1/2}; \qquad z \in M.$$

Note that a polynomial $y-m$ has at most $n-r$ zeroes in $T \backslash S$. Thus M is a Haar subspace on $T \backslash S$ of dimension $n+1-r$. This in conjunction with the fact that $t_j \in T \backslash S$ enables us to apply Theorem 6.3 from [9] in order to show that $k \geqslant n+2-r$. Consequently, the seminorm $|\cdot|_m$ is a norm on M. Hence there exists a constant $c>0$ such that $|z|_m^2 \geqslant c\,\|z\|^2$ for all $z=y-m$ in M, which completes the proof. ▮

Finally, Theorems 1.1 and 4.1 yield

COROLLARY 4.1. *Let x be a function in $C(T)$, and let $c>0$ be defined as in Theorem* 4.1. *Then the metric projection P: $C(T) \to M[x]$ satisfies the local Hölder condition*

$$\rho(Px, Pz) \leqslant d\,\|x-z\|^{1/2}$$

for all $z \in C(T)$ such that $\|z\| \leqslant K$, where K is an arbitrary positive constant and the constant d is defined as in Corollary 3.1.

The Hausdorff metric $\rho(Px, Pz)$ in the corollary can be replaced by $\|Px-Pz\|$ only if Pz is a one-element set. In particular, by Theorem 4.1 this is possible when $M[x]=M[z]$, i.e., when $x(s_i)=z(s_i)$ for $i=1, ..., p$. In general this is false even in the case of approximation by real algebraic polynomials with Lagrange interpolatory constraints. For example, let $x(t)=|t|$ $(-1 \leqslant t \leqslant 1)$, and let $M[x] \subset C[-1, 1]$ be defined by

$$M[x] = \{y \in \pi_1 : y(0)=x(0)\}.$$

Then we have $Pz=\{\alpha t\colon -1 \leqslant \alpha \leqslant 1\}$ for $z(t)=1-|t|$.

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